FINAL: ALGEBRA I

Date: **9th September 2025**
The Total points is **110** and the maximum you can score is **100** points.

(1) (4+11=15 points) Let G be a set, $e \in G$ and $*$ be a binary operator on G. When is $(G, *, e)$ called a group? Let $f : \mathbb{N} \to \mathbb{Z}$ be the bijective function given by $f(m) = m/2$ if m is even and $f(m) = (1-m)/2$ if m is odd. Define the binary operator on $\mathbb{N}$ as follows: For $a, b \in \mathbb{N}$, $a * b = f^{-1}(f(a) + f(b))$. Show that $(\mathbb{N}, *, 1)$ is an abelian group.

(2) (6+6+6+6=24 points) Prove or disprove
  (a) The groups $(\mathbb{C}, +)$ and $(\mathbb{R}, +)$ are isomorphic.
  (b) Let G be a group. If x and y are elements of finite order then xy is of finite order.
  (c) Let G and H be groups. Then $Z(G \times H) = Z(G) \times Z(H)$ where $Z(G)$ denotes the center of G.
  (d) Let G be a group, $H \trianglelefteq G$ and $K \trianglelefteq H$ then $K \trianglelefteq G$.

(3) (12 points) Find all group homomorphisms from $(\mathbb{Z}/12\mathbb{Z}, +)$ to D_{20}.

(4) (10 points) Compute all the conjugacy classes of the dihedral group D_{10} of order 10.

(5) (8+4=12 points) Show that the group $\mathbb{R}/\mathbb{Z}$ is isomorphic to $(\{z \in \mathbb{C} : |z| = 1\}$. Which elements are of finite order in $(\{z \in \mathbb{C} : |z| = 1\}$ and $\mathbb{Q}/\mathbb{Z}$?

(6) (3+14=17 points) Define a simple group. Show that a group of order 90 is not simple.

(7) (4+4+12=20 points) Let G be a group and A be a set. What does it mean to say that G acts on A? When is this action called transitive and when is it called doubly transitive? Show that the natural action of A_n on the set $\{1, 2, \ldots, n\}$ given by $\sigma \cdot i = \sigma(i)$ for $\sigma \in A_n$ and $1 \leq i \leq n$ is transitive for $n \geq 3$ and doubly transitive for $n \geq 4$.